Subject: Leaving Certificate Maths

Teacher: Mr Murphy

Lesson 21: Induction & Logs

21.1 Learning Intentions

After this week's lesson you will be able to;

- Explain what proof by induction is and where it can be used
- Complete a proof by induction
- Explain the relationship between logs and indices
- Rearrange equations with indices into logs

21.2 Specification

– prove by induction
  - simple identities such as the sum of the first n natural numbers and the sum of a finite geometric series
  - simple inequalities such as $n! > 2^n$, $2^n \geq n^2$ $(n \geq 4)$ $(1+x)^n \geq 1+nx$ $(x > -1)$
  - factorisation results such as 3 is a factor of $4^n - 1$

– solve problems using the rules of logarithms
  - $\log_a(xy) = \log_a x + \log_a y$
  - $\log_a(\frac{x}{y}) = \log_a x - \log_a y$
  - $\log_a x^q = q\log_a x$
  - $\log_a a = 1$ and $\log_a 1 = 0$
  - $\log_a x = \frac{\log_b x}{\log_b a}$

21.3 Chief Examiner's Report

These topics did not appear on the 2015 exam and thus, there are no comments in the CER.

21.4 Proof by Induction

Mathematical induction is a method of proof used to show that a given statement is true for all natural numbers.

It is done by proving that the first statement in an infinite sequence is true.

Next, we prove that if any one statement in the infinite sequence is true, then so is the next one.

In mathematical terms that means:

1) We prove the statement is true for n = 1.
2) Assume true for n = k.
3) Show true for n = k + 1.
4) If true for n = k then also for n = k+ 1, therefore true for all values of n

This method can be used under three headings:

Division, Series and Inequalities.
Question (Series):

i) Prove by induction that for any n, the sum of the first n natural numbers is $\frac{n(n+1)}{2}$

Question (Division):

ii) Prove by induction that $8^n - 1$ is divisible by 7 for all $n \in \mathbb{N}$

Question (Inequality):

iii) Prove, using induction, that $f(n) \geq g(n)$, *where* $n \geq 2$ *and* $n \in \mathbb{N}$.

21.5 Logarithms

We often look at logarithms (logs) as simply an inverse operation to exponents and leave the understanding in the abstract. However, if we look at what they really mean, they are actually quite simple.

For example, how many 5s do I multiply to get 125? The answer here is 3. The 3 is referred to as the logarithm or log.

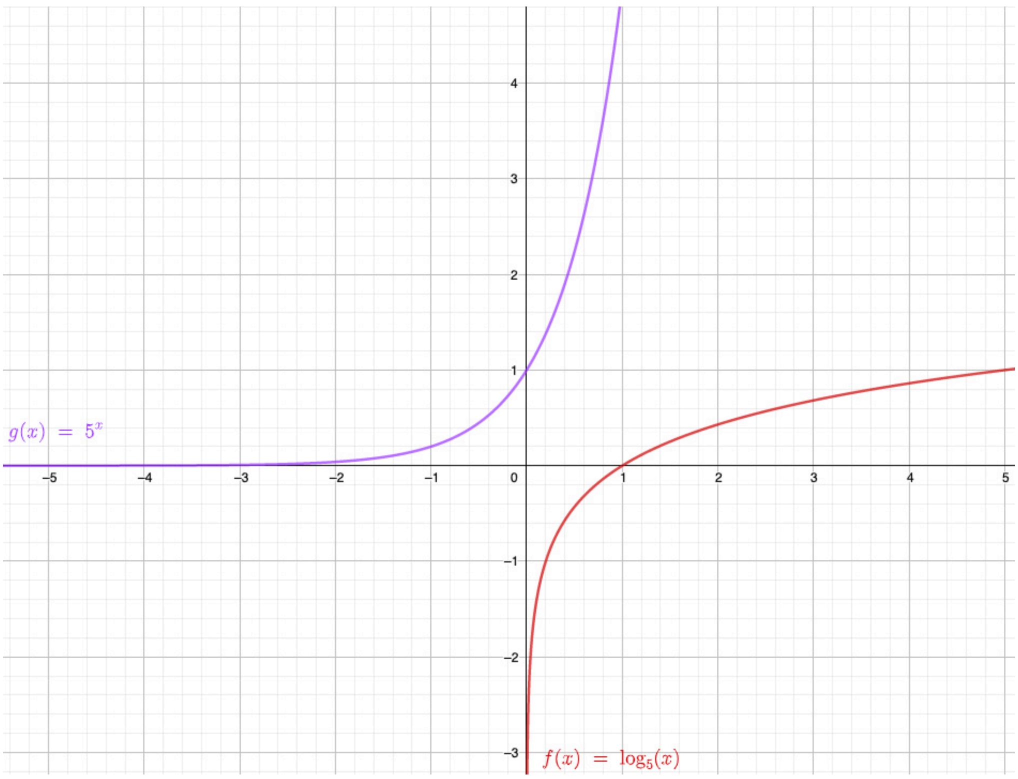
In mathematical language that looks like this:

$$\log_5 125 = 3$$

Through looking at this closely we can see that we can rewrite this equation in another way

$$5^3 = 125$$

This connection is one we often exploit when dealing with exponents and logs. Let's have a look at a visual representation:



In working with logs, we have plenty some laws that can help with the manipulation:

1) $\log_a xy = \log_a x + \log_a y$

2) $\log_a \frac{x}{y} = \log_a x - \log_a y$

3) $\log_a x^n = n\log_a x$

4) $\log_a a = 1$

5) $\log_a 1 = 0$

6) $\log_a x = \frac{\log_b x}{\log_b a}$

Questions:

1) $\log_2 (x - 2) + \log_2 x = 3$

2) $4\log_x 2 - \log_2 x - 3 = 0$

Sometimes it is possible to predict the future population in a city using a function.
The population in Sapphire City, over time, can be predicted using the following function:

$$p(t) = Se^{0\cdot 1t} \times 10^6.$$

The population in Avalon, over time, can be predicted using the following function:

$$q(t) = 3\cdot 9e^{kt} \times 10^6.$$

In the functions above, t is time, in years; $t = 0$ is the beginning of 2010; and both S and k are constants.

(a) The population in Sapphire City at the beginning of 2010 is 1 100 000 people.
Find the value of S.

(b) Find the predicted population in Sapphire City at the beginning of 2015.

(c) Find the predicted change in the population in Sapphire City during 2015.

(d) The predicted population in Avalon at the beginning of 2011 is 3 709 795 people.
Write down and solve an equation in k to show that $k = -0{\cdot}05$, correct to 2 decimal places.

(e) Find the year during which the populations in both cities will be equal.

(f) Find the predicted average population in Avalon from the beginning of 2010 to the beginning of 2025.

(g) Use the function $q(t) = 3{\cdot}9e^{-0{\cdot}05t} \times 10^6$ to find the predicted rate of change of the population in Avalon at the beginning of 2018.

21.6 Recap of the Learning Intentions

After this week's lesson you will be able to;

- Explain what proof by induction is and where it can be used
- Complete a proof by induction
- Explain the relationship between logs and indices
- Rearrange equations with indices into logs

21.7 Homework Task

Prove by induction that $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$ for any $n \in \mathbb{N}$

21.8 Solutions to 20.12

b. Alex has a credit card debt of €5000. One method of clearing this debt is to make a fixed repayment at the end of each month. The amount of this repayment is 2.5% of the original debt.

(i) What is the fixed monthly repayment, €A, required to pay the debt of €5000?

$$2.5\% \times €5000$$

(ii) The annual percentage rate (APR) charged on debt by the credit card company is 21.75%, fixed for the term of the debt. Find as a percentage, correct to 3 significant figures, the monthly interest rate that is equivalent to an APR of 21.75%.

$$(1 \times i)^{\frac{1}{12}} = (1.2175)^{\frac{1}{12}} = 1.016535 = 1.65\%$$

(iii) Assume Alex pays the fixed monthly repayment, €A, each month and does not have any further transactions on that card. Complete the table below to show how the balance of the debt of €5000 is reducing each month for the first three months, assuming an APR of 2175%, charged and compounded monthly.

Payment number	Fixed monthly payment, €A	€A		New balance of debt (€)
		Interest	Previous balance reduced by (€)	
0				5000
1	125	82.50	42·50	4957·50
2	125	81.80	43.20	4914.30
3	125	81.09	43.91	4870.39

(iv) Using the formula you derived on the previous page, or otherwise, find how long it would take to pay off a credit card debt of €5000, using the repayment method outlined at the beginning of **part (b)** above. Give your answer in months, correct to the nearest month.

$$\textit{Rearrange the ammortisation formula to get}$$

$$\frac{A}{A-pi} = (1+t)^t$$

$$\frac{125}{125-5000\frac{1.65}{100}} = (1+\frac{1.65}{100})^t$$

$$\frac{125}{42.5} = (1.0165)^t$$

$$\log\left(\frac{125}{42.5}\right) = t\log 1.0165$$

$$t = \frac{\log\left(\frac{125}{42.5}\right)}{\log 1.0165}$$

$$t = 66\ months$$

(v) Alex decides to borrow €5000 from the local Credit Union to pay off this credit card debt of €5000. The APR charge for the Credit Union loan is 8.5% fixed for the term of the loan. The loan is to be repaid in equal weekly repayments, at the end of each week, for 156 weeks. Find the amount of each weekly repayment.

$$A = \frac{pi(1+i)^t}{(1+i)^t-1}$$

$$A = \frac{5000\left(1.085^{\frac{1}{52}}-1\right)(1.085)^3}{(1.085)^3-1}$$

$$A = €36.16$$

(vi) How much will Alex save by paying off the credit card debt using the loan from the Credit Union instead of paying the fixed repayment from **part (b)(i)** each month to the credit card company?

$$125 \times 66 - (36.16)(156) = €2609.04$$